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Dynamical and Geometric Phases of Bose-Einstein Condensates

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Abstract By using of the Lewis-Riesenfeld invariant theory, dynamical and geometric phases of Bose-Einstein condensates are studied. The Aharonov-Anandan phase is also obtained under the cyclical evolution.

Keywords Phase · Bose-Einstein condensate

1 Introduction

Recently, much attention has been paid to the investigation of Bose-Einstein condensation (BEC) in dilute and ultracold gases of neutral alkali-metal atoms using a combination of laser and evaporative cooling [1–3] due to the optical properties [4, 5], statistical properties [6, 7], phase properties [8, 9], and tunneling effect [10–18].

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld [19] is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in [20] by introducing the concept of basic invariants and used to study the geometric phases [21, 22] in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations [23, 24], but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions [25–32]. In this paper, by using of the Lewis-Riesenfeld invariant theory, we shall study dynamical and the geometric phases of Bose-Einstein condensates.

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2 Model

The two internal states of the BEC are coupled by a near resonant pulsed radiation field [33]. Total density and mean phase remain constant during the condensate evolution, then Hamiltonian describing the transition between the two internal states reads [34]

$$H = \mu(a_1^\dagger a_1 - a_2^\dagger a_2) + g(a_1^\dagger a_1 - a_2^\dagger a_2)^2 + K\delta_T(t)(a_1^\dagger a_2 + a_2^\dagger a_1), \quad (1)$$

where K is the coupling strength between the two internal states, g is the interaction, and μ is the difference between the chemical potentials of two components. a_i $(a_i^\dagger)$ $(i = 1, 2)$ are boson annihilation (creation) operators for the two components. $\delta_T(t) = \sum_n \delta(t - nT)$ means that the radiation field is only turned on at certain discrete moments, namely, integral multiples of the period T.

3 Dynamical and Geometric Phases of Bose-Einstein Condensates

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory [19]. For a one-dimensional system whose Hamiltonian $H(t)$ is time-dependent, then there exists an operator $I(t)$ called invariant if it satisfies the equation

$$i\frac{\partial I(t)}{\partial t} + [I(t), H(t)] = 0. \quad (2)$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$I(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \quad (3)$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = H(t)|\psi(t)\rangle_s. \quad (4)$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of (4) is different from the eigenfunction $|\lambda_n, t\rangle$ of $I(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \quad (5)$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of (4). Then the general solution of the Schrödinger equation (4) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \quad (6)$$

where

$$\delta_n(t) = \int_0^t dt' \left\langle \lambda_n, t' \left| i\frac{\partial}{\partial t'} - H(t') \right| \lambda_n, t' \right\rangle, \quad (7)$$

and $C_n = \langle\lambda_n, 0|\psi(0)\rangle_s$.

It is easy to find that $I_1(t) = N_1^2 + N_2^2 + 2N_1N_2$ (where $N_i = a_i^\dagger a_i$ for $i = 1, 2$) is a special invariant of this system and satisfies $\hat{I}_1(t)|m\rangle_{a_1}|n\rangle_{a_2} = \lambda_{mn}|m\rangle_{a_1}|n\rangle_{a_2}$, where $a_1^\dagger a_1|m\rangle_{a_1} = m|m\rangle_{a_1}$, $a_2^\dagger a_2|n\rangle_{a_2} = n|n\rangle_{a_2}$, and $\lambda_{mn} = m^2 + n^2 + 2mn$.

In the following, we can restrict the space being in the sub-space of the eigenstate of the invariant $I_1(t)$. Corresponding, $I_1(t)$ appearing in (1) can be replaced by its eigenvalue λ_{mn}. Correspondingly, (1) becomes

$$H = \mu(a_1^\dagger a_1 - a_2^\dagger a_2) + g\lambda_{mn} + K\delta_T(t)(a_1^\dagger a_2 + a_2^\dagger a_1) - 4ga_1^\dagger a_1 a_2^\dagger a_2. \tag{8}$$

In order to obtain the exact solutions of (4), we can define operators K_+, K_- and K_0 as follows:

$$K_+ = a_1^\dagger a_2, \qquad K_- = a_2^\dagger a_1, \qquad K_0 = a_1^\dagger a_1 - a_2^\dagger a_2, \tag{9}$$

which hold the commutation relations

$$[K_0, K_\pm] = \pm 2K_\pm, \qquad [K_+, K_-] = K_0, \tag{10}$$

it is easy to prove that operators K_+, K_- and K_0 together with the Hamiltonian H construct a quasi-algebra.

Then we can get the L-R invariant as follows

$$I_2(t) = \cos\theta K_0 - e^{-i\varphi}\sin\theta K_+ - e^{i\varphi}\sin\theta K_-, \tag{11}$$

it is apparent that $[I_1(t), I_2(t)] = 0$. Here θ and φ are determined by the equation $i\partial I_2(t)/\partial t + [I_2(t), H(t)] = 0$, and satisfy the relations

$$\dot\theta = 4K\delta_T(t)\sin\varphi, \tag{12}$$

$$\dot\varphi\sin\theta\sin\varphi - \dot\theta\cos\theta\cos\varphi - 2\mu\sin\theta\sin\varphi = 0, \tag{13}$$

$$2K\delta_T(t)\cos\theta + 2\mu\sin\theta\cos\varphi - \dot\varphi\sin\theta\cos\varphi - \dot\theta\cos\theta\sin\varphi = 0, \tag{14}$$

where dot denotes the time derivative, and we have adopted an approximation, i.e., $N_1 \gg 1$, $N_2 \gg 1$, and $N_1 \approx N_2$.

According to the unitary transformation method, we can construct the unitary transformation

$$V(t) = \exp[\sigma K_+ - \sigma^* K_-], \tag{15}$$

where $\sigma = \frac{\theta}{2}e^{-i\varphi}$ and $\sigma^* = \frac{\theta}{2}e^{i\varphi}$. The invariant $I_2(t)$ can be transformed into a new time-independent operator I_V:

$$I_V = V^\dagger(t) I_2(t) V(t) = K_0. \tag{16}$$

Correspondingly, we can get the eigenvalue equation of operator $I(t)$

$$\hat I_V |m\rangle_{a_1}|n\rangle_{a_2} = (m-n)|m\rangle_{a_1}|n\rangle_{a_2}. \tag{17}$$

In terms of the unitary transformation $V(t)$ and the Baker-Campbell-Hausdorff formula [35]

$$V^\dagger(t)\frac{\partial V(t)}{\partial t} = \frac{\partial L}{\partial t} + \frac{1}{2!}\left[\frac{\partial L}{\partial t}, L\right] + \frac{1}{3!}\left[\left[\frac{\partial L}{\partial t}, L\right], L\right] + \frac{1}{4!}\left[\left[\left[\frac{\partial L}{\partial t}, L\right], L\right], L\right] + \cdots, \tag{18}$$

where $V(t) = \exp[L(t)]$, one has

$$\begin{aligned} H_V(t) &= V^\dagger(t)H(t)V(t) - iV^\dagger(t)\frac{\partial V(t)}{\partial t} \\ &= \left(g + \frac{1}{2}\sin^2\theta\right)\lambda_{mn} \\ &\quad + \left[\mu\cos\theta - K\delta_T(t)\sin\theta\cos\varphi - \frac{1}{2}\sin^2\theta + \frac{\dot\varphi}{2}(1-\cos\theta)\right]a_1^\dagger a_1 \\ &\quad - \left[\mu\cos\theta - K\delta_T(t)\sin\theta\cos\varphi + \frac{1}{2}\sin^2\theta + \frac{\dot\varphi}{2}(1-\cos\theta)\right]a_2^\dagger a_2 \\ &\quad + (2-3\sin^2\theta)a_1^\dagger a_1 a_2^\dagger a_2, \end{aligned}$$

where λ_{mn} is the eigenvalue of operator $I_1(t)$ given above. In (19), we have let the coefficients of $a_1^\dagger a_2$ and $a_2^\dagger a_1$ equal to zero under the following conditions

$$\mu\sin\theta\cos\varphi + K\delta_T(t)\left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\cos 2\varphi\right) = 0, \tag{19}$$

$$K\delta_T(t)\sin^2\frac{\theta}{2}\sin 2\varphi - \mu\sin\theta\sin\varphi = 0. \tag{20}$$

It is easy to find that $H(t)$ differs from I_V only by a time-dependent c-number factor. Thus we can get the general solution of the time-dependent Schrödinger equation (4)

$$|\Psi(t)\rangle_s = \sum_n\sum_m C_{nm}\exp[i\delta_{nm}(t)]\hat V(t)|m\rangle_{a_1}|n\rangle_{a_2}, \tag{21}$$

with the coefficients $C_{nm} = \langle n, m, t=0|\Psi(0)\rangle_s$. The phase $\delta_{nm}(t) = \delta^d_{nm}(t) + \delta^g_{nm}(t)$ includes the dynamical phase

$$\begin{aligned} \delta^d_{nm}(t) &= -m\int_{t_0}^t\left[\mu\cos\theta - K\delta_T(t)\sin\theta\cos\varphi - \frac{1}{2}\sin^2\theta\right]dt' - \int_{t_0}^t\left(g + \frac{1}{2}\sin^2\theta\right)\lambda_{mn}dt' \\ &\quad + n\int_{t_0}^t\left[\mu\cos\theta - K\delta_T(t)\sin\theta\cos\varphi + \frac{1}{2}\sin^2\theta\right]dt' - mn\int_{t_0}^t(2-3\sin^2\theta)dt', \end{aligned} \tag{22}$$

and the geometric phase

$$\delta^g_{nm}(t) = \int_{t_0}^t(n-m)\frac{\dot\varphi}{2}(1-\cos\theta)dt'. \tag{23}$$

Particularly, the geometric phase becomes under the cyclical evolution

$$\delta^g_{nm}(t) = \frac{1}{2}\oint(n-m)(1-\cos\theta)d\varphi, \tag{24}$$

which is the known geometric Aharonov-Anandan phase.

4 Conclusions

In conclusion, by using of the L-R invariant theory, we have studied phases of Bose-Einstein condensates, dynamical and geometric phases are presented respectively. The Aharonov-Anandan phase is obtained under the cyclical evolution.

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